

Symplectic forms on the space of embedded symplectic surfaces and their reductions

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Abstract

Let (M, ω) be a symplectic manifold, and (Σ, σ) a closed connected symplectic 2-manifold. We construct a weakly symplectic form $\omega^D_{(\Sigma, \sigma)}$ on the space of immersions $\Sigma \rightarrow M$ that is a special case of Donaldson's form. We show that the restriction of $\omega^D_{(\Sigma, \sigma)}$ to any orbit of the group of Hamiltonian symplectomorphisms through a symplectic embedding $(\Sigma, \sigma) \hookrightarrow (M, \omega)$ descends to a weakly symplectic form ω^D_{red} on the quotient by $\text{Symp}(\Sigma, \sigma)$, and that the obtained symplectic space is a symplectic quotient of the subspace of symplectic embeddings $\mathcal{S}_e(\Sigma, \sigma)$ with respect to the $\text{Symp}(\Sigma, \sigma)$ -action. We also compare $\omega^D_{(\Sigma, \sigma)}$ and its reduction ω^D_{red} to another 2-form on the space of immersed symplectic Σ -surfaces in M . We conclude by a result on the restriction of $\omega^D_{(\Sigma, \sigma)}$ to moduli spaces of J -holomorphic curves.

1 Introduction

Let (M, ω) be a compact finite-dimensional symplectic manifold, and Σ a closed connected 2-manifold. Fix a symplectic form σ on Σ . We identify the tangent space to $C^\infty(\Sigma, M)$ at $f: \Sigma \rightarrow M$ with the space $\Omega^0(\Sigma, f^*(TM))$ of smooth vector fields $\tau: \Sigma \rightarrow f^*(TM)$.

Definition 1.1 Define a 2-form on $C^\infty(\Sigma, M)$ by

$$(\omega^D_{(\Sigma, \sigma)})_f(\tau_1, \tau_2) := \int_{\Sigma} \omega_{f(x)}(\tau_1(x), \tau_2(x)) \sigma,$$

where $\tau_1, \tau_2 \in T_f(C^\infty(\Sigma, M))$. ⊗

The form $\omega^D_{(\Sigma, \sigma)}$ is a special case of the two-form on the space of smooth maps $S \rightarrow M$ of a compact oriented manifold S equipped with a fixed volume form η , introduced by Donaldson in [3]. Under some topological conditions, e.g., that $H^1(S) = 0$ and for all $i \in C^\infty(\Sigma, M)$ the class $i^*[\omega]$ is the zero class in $H^2(S)$, Donaldson described a moment map for the action of the Lie group of volume preserving diffeomorphisms $\text{Diff}(S, \eta)$ on $C^\infty(S, M)$. This action restricts to a Hamiltonian action on the subspace of embeddings $\text{Emb}(S, M)$. In [6], Brian Lee gives a rigorous formulation of Donaldson's heuristic construction, in the ‘‘Convenient Setup’’ of Frölicher, Kriegl, and Michor [5], and shows that the form reduces to the image of $\text{Ham}(M, \omega)$ -orbits through isotropic embeddings in $\text{Emb}(S, M)$ under the projection to the quotient $\text{Emb}(S, M)/\text{Diff}(S, \eta)$. Lee's result does not assume $H^1(S) = 0$. In this paper, we will omit also the condition $i^*[\omega] = 0$, and instead of looking at orbits through isotropic embeddings, we will look at orbits through symplectic embeddings. Denote by

$$\mathcal{S}_e(\Sigma, \sigma)$$

the subspace of symplectic embeddings $(\Sigma, \sigma) \rightarrow (M, \omega)$. The Lie group $\text{Symp}(\Sigma, \sigma)$ of diffeomorphisms of Σ that pull back σ to itself acts freely on $\mathcal{S}_e(\Sigma, \sigma)$ on the right. In this paper we study the reduction of $\omega^D_{(\Sigma, \sigma)}$ to $\mathcal{S}_e(\Sigma, \sigma)$ modulo $\text{Symp}(\Sigma, \sigma)$ and to moduli spaces of un-parametrized J -holomorphic curves. The terms smooth manifold and map, tangent space, and differential form are interpreted in the ‘‘Convenient Setup’’. In this framework, the local model is the convenient vector space: a locally convex vector space E with the property that for any smooth (infinitely differentiable) curve $c_1: \mathbb{R} \rightarrow E$ there is a curve $c_2: \mathbb{R} \rightarrow E$ such that $c_2' = c_1$, with the c^∞ -topology: the finest topology for which all smooth curves $\mathbb{R} \rightarrow E$ are continuous. (The c^∞ -topology is finer than the locally convex topology on E . If E is a Frechet space, (i.e., a complete and metrizable locally convex space), then the two topologies coincide.) A map between convenient vector spaces is smooth if it sends smooth curves to smooth curves. Smooth manifolds are modeled on convenient vector spaces via charts, whose transition functions are smooth; a map between smooth manifolds is smooth if it maps smooth curves to smooth curves. (See [5] and [6, Sec. 2].)

In the Appendix we show that the 2-form $\omega^D_{(\Sigma, \sigma)}$ is closed and its restriction to the space of immersions $\Sigma \rightarrow M$ is weakly non-degenerate. We also show that for an almost complex structure $J: TM \rightarrow TM$ that is compatible with ω , the induced almost complex structure $\tilde{J}: \text{TC}^\infty(\Sigma, M) \rightarrow \text{TC}^\infty(\Sigma, M)$ is compatible with $\omega^D_{(\Sigma, \sigma)}$. In Section 2 we prove that $\mathcal{S}_e(\Sigma, \sigma)$ is a smooth manifold and describe its tangent bundle, see Proposition 2.2; we show that the restriction of the form $\omega^D_{(\Sigma, \sigma)}$ to $\mathcal{S}_e(\Sigma, \sigma)$ is weakly symplectic, see Proposition 2.10. In Section 3 we prove the following theorem.

Theorem 1.2. *Let \mathcal{N} be a $\text{Ham}(M, \omega)$ -orbit in $\mathcal{S}_e(\Sigma, \sigma)$. The restriction of $\omega^D_{(\Sigma, \sigma)}$ to \mathcal{N} descends to a closed weakly non-degenerate 2-form ω^D_{red} on the image \mathcal{O} in the orbit space under the projection $q: \mathcal{S}_e(\Sigma, \sigma) \rightarrow \mathcal{S}_e(\Sigma, \sigma)/\text{Symp}(\Sigma, \sigma)$. The symplectic space $(\mathcal{O}, \omega^D_{\text{red}})$ is a symplectic quotient of $\mathcal{S}_e(\Sigma, \sigma)$ with respect to the $\text{Symp}(\Sigma, \sigma)$ -action.*

The notion of a symplectic quotient here does not depend on having a moment map, see Definition 3.11. It is motivated by the optimal reduction method of Ortega and Ratiu [11].

We also compare $\omega^D_{(\Sigma, \sigma)}$ to the 2-form we defined in [2] on the space of immersed symplectic Σ -surfaces in M . Denote by

$$\text{ev}: C^\infty(\Sigma, M) \times \Sigma \rightarrow M$$

the *evaluation map*

$$\text{ev}(f, x) := f(x).$$

Definition 1.3 Define a 2-form on $C^\infty(\Sigma, M)$ as the push-forward of the 4-form $\text{ev}^*(\omega \wedge \omega)$ along the coordinate-projection $\pi_{C^\infty(\Sigma, M)}: C^\infty(\Sigma, M) \times \Sigma \rightarrow C^\infty(\Sigma, M)$ by

$$(\omega_{C^\infty(\Sigma, M)})_f(\tau_1, \tau_2) := \int_{\{f\} \times \Sigma} \iota_{(\ell_1 \wedge \ell_2)} \text{ev}^*(\omega \wedge \omega). \quad (1.1)$$

Here $\ell_i \in T(C^\infty(\Sigma, M) \times \Sigma)$ is a *lifting* of $\tau_i \in T_f(C^\infty(\Sigma, M))$, i.e.,

$$d(\pi_{C^\infty(\Sigma, M)})_{\ell_i(f, x)} = \tau_i \text{ at each point } (f, x) \in \pi_{C^\infty(\Sigma, M)}^{-1}(f).$$

⊗

Denote

$$\mathcal{S}_i(\Sigma) := \{f: \Sigma \rightarrow M \mid f \text{ is an immersion, } f^*\omega \text{ is a symplectic form on } \Sigma\}.$$

The space $\mathcal{S}_i(\Sigma)$ is an open subset of $C^\infty(\Sigma, M)$ in the C^∞ -topology. Let

$$\omega_{\mathcal{S}_i(\Sigma)}$$

be the 2-form on $\mathcal{S}_i(\Sigma)$ given by the restriction of $\omega_{C^\infty(\Sigma, M)}$. We showed in [2] that the 2-form $\omega_{C^\infty(\Sigma, M)}$ on $C^\infty(\Sigma, M)$ is well defined and closed, and $\omega_{C^\infty(\Sigma, M)}(\tau, \cdot)$ vanishes at f if τ is everywhere tangent to $f(\Sigma)$. Furthermore,

$$\omega_{\mathcal{S}_i(\Sigma)}(\tau, \cdot) = 0 \text{ at } f \iff \tau \text{ is tangent to } f(\Sigma) \text{ at every } x \in \Sigma.$$

We say that a vector field $\tau: \Sigma \rightarrow f^*(TM)$ is *tangent to $f(\Sigma)$ at x* if $\tau(x) \in df_x(T_x \Sigma)$. See also [7].

Consider the space of ω -compatible almost complex structures $\mathcal{J} = \mathcal{J}(M, \omega)$ on (M, ω) . Fix $\Sigma = (\Sigma, j)$, where j is a complex structure on Σ . The moduli space $\mathcal{M}_i(A, \Sigma, J)$ is the space of simple immersed (j, J) -holomorphic Σ -curves in a homology class $A \in H_2(M, \mathbb{Z})$. The moduli space $\mathcal{M}_e(A, \Sigma, J)$ is the space of embedded (j, J) -holomorphic Σ -curves in a homology class $A \in H_2(M, \mathbb{Z})$. We look at almost complex structures that are regular for the projection map

$$p_A: \mathcal{M}_i(A, \Sigma, \mathcal{J}) \rightarrow \mathcal{J};$$

for such a J , the spaces $\mathcal{M}_i(A, \Sigma, J)$ and $\mathcal{M}_e(A, \Sigma, J)$ are finite-dimensional manifolds. (The set of p_A -regular ω -compatible almost complex structures is of the second category in \mathcal{J} .) See [9, Thm 3.1.5]. There is merit to the form $\omega_{\mathcal{S}_i(\Sigma)}$ in the fact that it is degenerate along directions tangent to $f(\Sigma)$, hence descends to a well defined form on the quotient space $\widetilde{\mathcal{M}}_i(A, \Sigma, J)$ of $\mathcal{M}_i(A, \Sigma, J)$ by the proper action of the group $\text{Aut}(\Sigma, j)$ of bi-holomorphisms of Σ : this enables us to apply Gromov's compactness theorem and get a well defined invariant of (M, ω) . If $J_* \in \mathcal{J}_{\text{reg}}(A)$ is integrable, then the restriction of the form $\omega_{\mathcal{S}_i(\Sigma)}$ to $\mathcal{M}_i(A, \Sigma, J_*)$ is non-degenerate, up to reparametrizations; see [2, Prop. 4.4]. We obtained results on the existence of J -holomorphic curves in a homology class A for some subset of \mathcal{J} , and in some cases for a generic J , see [2, Cor. 1.3].

Here we show that the 2-forms $2\omega_{(\Sigma, \sigma)}^D$ and $\omega_{\mathcal{S}_i(\Sigma)}$ coincide in exact direction, hence on the quotient of a $\text{Ham}(M, \omega)$ -orbit with respect to the $\text{Symp}(\Sigma, \sigma)$ -action. The difference between ω^D and $\omega_{\mathcal{S}_i(\Sigma)}$ is that $\iota_v \omega^D$ is degenerate along vectors in $\text{TS}_e(\Sigma, \sigma)$ everywhere tangent to Σ iff $v = i^* V_H$ for a Hamiltonian vector field V_H on M whereas $\iota_v \omega_{\mathcal{S}_i(\Sigma)}$ is degenerate along vectors everywhere tangent to Σ for every v in $\text{TS}_i(\Sigma)$, see Remark 3.14. Due to this difference, Theorem 1.2 does not hold for $\omega_{\mathcal{S}_i(\Sigma)}$. On the other hand, we do not get a well defined reduction of $\omega_{(\Sigma, \sigma)}^D$ on the quotient of $\mathcal{M}_i(A, \Sigma, J)$ by the action of $\text{Aut}(\Sigma, j)$, as we did for $\omega_{\mathcal{S}_i(\Sigma)}$. However we do get a partial result. For $J \in \mathcal{J}(M, \omega)$, denote by

$$\text{Ham}^J(M, \omega)$$

the subgroup of $\text{Ham}(M, \omega)$ of J -holomorphic Hamiltonian symplectomorphisms. Let \mathcal{N} be an orbit of $\text{Ham}^J(M, \omega)$ through an embedded (j, J) -holomorphic curve $f: \Sigma \rightarrow M$ for which $f^* \omega = \sigma$. The orbit \mathcal{N} is a subset of $\mathcal{M}_e(A, \Sigma, J)$, where

$$A \in H_2(M, \mathbb{Z})$$

is the class for which the area $f^* \omega(\Sigma) = \sigma(\Sigma)$ for (every) $f \in A$.

Corollary 1.4. *Assume that the symplectic form σ on Σ is compatible with the complex structure j on Σ . Let $J \in \mathcal{J}(M, \omega)$, assume that J is integrable and regular for A . Let \mathcal{N} be an orbit of $\text{Ham}^J(M, \omega)$ through a (j, J) -holomorphic embedding $f: \Sigma \rightarrow M$ for which $f^* \omega = \sigma$.*

The forms $\omega_{(\Sigma, \sigma)}^D$ and $\omega_{\mathcal{S}_i(\Sigma)}$ descend to well defined symplectic forms ω_{red}^D and $\omega_{\mathcal{S}_i(\Sigma)}^{\text{red}}$ on the quotient of \mathcal{N} with respect to $\text{Aut}(\Sigma, j)$. The form $\omega_{\mathcal{S}_i(\Sigma)}^{\text{red}}$ coincides with the form $2\omega_{\text{red}}^D$ on the quotient.

2 The space $\mathcal{S}_e(\Sigma, \sigma)$

Proposition 2.1. *The 2-form $\omega^D_{(\Sigma, \sigma)}$ on $C^\infty(\Sigma, M)$ is closed and its restriction to the space of immersions $\Sigma \rightarrow M$ is weakly non-degenerate.*

The space $C^\infty(\Sigma, M)$ is a smooth manifold in the Convenient Setup, modeled on spaces $\Gamma(f^*TM)$ of sections of the pullback bundle along $f \in C^\infty(\Sigma, M)$ [5, 42.1]. The space $\Gamma(f^*TM)$ has a natural convenient structure [5, 30.1].

A 2-form Ω on a manifold X (possibly infinite-dimensional) is called *weakly non-degenerate* if for every $x \in X$ and $0 \neq v \in T_x X$ there exists a $w \in T_x X$ such that $\Omega_x(v, w) \neq 0$. This is equivalent to its associated vector bundle homomorphism $\Omega^\flat: TX \rightarrow T^*X$ being injective. If $\Omega^\flat: TX \rightarrow T^*X$ is an isomorphism, i.e., invertible with a smooth inverse, then Ω is called *strongly non-degenerate*. In this paper, by non-degenerate we mean weakly non-degenerate. If Ω is closed and weakly non-degenerate, it is called *weakly symplectic*.

For the proof of Proposition 2.1 and required facts on compatible almost complex structures, see the Appendix.

Notation:

For every embedding $i: \Sigma \rightarrow M$, for $v \in \Gamma(i^*TM)$, let $\alpha_v \in \Omega^1(\Sigma)$ denote the form

$$(\alpha_v)_x(\xi) := \omega_x(v(x), di_x \xi) \text{ for } \xi \in T_x \Sigma.$$

Also, set

$$\Gamma_{\text{closed}}(i^*TM) := \{v \in \Gamma(i^*TM) \mid \alpha_v \text{ is a closed 1-form on } \Sigma\},$$

and

$$\Gamma_{\text{exact}}(i^*TM) := \{v \in \Gamma(i^*TM) \mid \alpha_v \text{ is an exact 1-form on } \Sigma\}.$$

For a vector field

$$v \in T_i C^\infty(\Sigma, M)$$

denote by

$$\xi_v + \tau_v \tag{2.2}$$

the decomposition of v to a vector field ξ_v everywhere ω -orthogonal to Σ and a vector field τ_v everywhere tangent to Σ . Such a decomposition exists and is unique, e.g., by Remark A.9 and Corollary A.8.

We say that a vector field $\tau: \Sigma \rightarrow TM$ is *tangent to Σ at x* if $\tau(x) \in T_{i(x)} i(\Sigma)$. We say that a vector field $\xi: \Sigma \rightarrow TM$ is *ω -orthogonal to Σ at x* if $\xi(x) \in (T_{i(x)} i(\Sigma))^\omega$.

Denote by

$$\mathcal{S}_e(\Sigma, \sigma)$$

the set of embeddings $(\Sigma, \sigma) \rightarrow (M, \omega)$ such that $i^*\omega = \sigma$.

Proposition 2.2. *The set $\mathcal{S}_e(\Sigma, \sigma)$ is a smooth manifold modeled on $\Gamma_{\text{exact}}(i^*TM) \oplus \mathcal{X}(\Sigma, \sigma)$, where*

$$\mathcal{X}(\Sigma, \sigma) = \{\xi \text{ a vector field on } \Sigma \mid \mathcal{L}_\xi \sigma = 0\}.$$

To prove the proposition, we first recall the symplectic tubular neighbourhood Theorem of Weinstein.

2.3 Consider a symplectic embedding $i: (\Sigma, \sigma) \hookrightarrow (M, \omega)$. The symplectic normal bundle

$$N\Sigma = \{(x, v) \mid x \in \Sigma, v \in T_{i(x)} M / T_x i(\Sigma)\} \rightarrow \Sigma.$$

The *minimal coupling form*, due to Sternberg [12], is a closed 2-form $\omega_{N\Sigma}$ with the following properties:

1. Its pullback to the fibers coincide with the fiberwise symplectic forms.
2. Its pullback to the zero section coincides with σ .
3. At the points of the zero section, the fibers of $N\Sigma$ are $\omega_{N\Sigma}$ -orthogonal to the zero section.

Consequently, $\omega_{N\Sigma}$ is non-degenerate near the zero section.

The symplectic normal bundle $N\Sigma$ can be realized as a subbundle of TM : the symplectic orthocomplement of $T\Sigma = Ti(\Sigma)$ in $TM|_{i(\Sigma)}$. In other words, the fiber $N_x\Sigma$ at $x \in \Sigma$ is identified with

$$(T_{i(x)} i(\Sigma))^\omega = \{v \in T_{i(x)} M \mid \omega(v, w) = 0 \text{ for every } w \in T_{i(x)} i(\Sigma)\}$$

with the symplectic form $\omega|_{(T_{i(x)} i(\Sigma))^\omega}$.

By the classical tubular neighbourhood theorem in differential topology combined with a theorem of Weinstein [13, Theorem 4.1], there exists a neighbourhood U of the zero section in $N\Sigma$ and a symplectic open embedding

$$\Phi_i: (U, \omega_{N\Sigma}) \rightarrow (M, \omega) \quad (2.3)$$

whose restriction to the zero section is i , and whose differential is di at every point of Σ . \circlearrowright

Lemma 2.4. *Let $\Sigma = (\Sigma, \sigma) \xrightarrow{i} (M, \omega)$ be an embedded closed connected symplectic submanifold of dimension 2. Let $v \in T_i C^\infty(\Sigma, M)$. If v is everywhere ω -orthogonal to Σ , then v equals the restriction i^*V_H to $i(\Sigma)$ of a Hamiltonian vector field V_H on M .*

Proof. By §2.3, we can consider ξ_v as a vector field ξ_0 on the zero section in $N\Sigma$; it is enough to show that ξ_0 extends to a Hamiltonian vector field ξ on a neighbourhood of the zero section in $N\Sigma$, since then the push forward of ξ via Φ_i in (2.3) is a Hamiltonian vector field on a neighbourhood of Σ in M . Then ξ_v can be extended to a Hamiltonian vector field on M , using a cut-off function with a support that is close enough to Σ .

By assumption, for every x in the zero section, $\xi_0(x)$ is in $N_x\Sigma = (T_{i(x)} i(\Sigma))^\omega$. Each of the fibers

$$((T_{i(x)} i(\Sigma))^\omega, \omega_{N\Sigma}|_{(T_{i(x)} i(\Sigma))^\omega}) = ((T_{i(x)} i(\Sigma))^\omega, \omega|_{(T_{i(x)} i(\Sigma))^\omega})$$

is a symplectic vector space; to each vector $\xi_0(x) \in (T_{i(x)} i(\Sigma))^\omega$ there corresponds a linear function $\omega_{N\Sigma}(\xi_0(x), \cdot) = \omega(\xi_0(x), \cdot)$ from the fiber to \mathbb{R} . Taking the union of these functions over the points of the zero section, we get a function

$$H: N\Sigma \rightarrow \mathbb{R} \quad (2.4)$$

which is smooth in a neighbourhood of the zero section in $(N\Sigma, \omega_{N\Sigma})$. Take ξ to be the vector field defined by

$$dH = \omega_{N\Sigma}(\xi, \cdot)$$

in a neighbourhood of the zero section on which $\omega_{N\Sigma}$ is non-degenerate. \square

Lemma 2.5. *Let $\Sigma = (\Sigma, \sigma) \xrightarrow{i} (M, \omega)$ be an embedded closed connected symplectic submanifold of dimension 2. Let $v \in \Gamma_{\text{closed}}(i^*TM)$. The following are equivalent.*

1. The vector field v equals the restriction i^*V_H to $i(\Sigma)$ of a Hamiltonian vector field V_H on M .
2. The form $\alpha_v = \omega(v, di(\cdot))$ on Σ is exact.

3. $(\omega^D_{(\Sigma, \sigma)})_i(v, w) = 0$ for every w that is everywhere tangent to Σ and satisfies $\mathcal{L}_{(di)^{-1}w}\sigma = 0$.
4. v is everywhere ω -orthogonal to Σ .

Recall that a vector field X on M is Hamiltonian if the form $\iota_X\omega$ is exact. Here $di: T\Sigma \rightarrow T(i(\Sigma))$.

Proof.

1 \Rightarrow 2 If V_H is a Hamiltonian vector field on M , then on Σ the form $\alpha_{i^*V_H}$ equals dh with $h = H \circ i$.

2 \Rightarrow 3 If $\alpha_v = dh$ for a function $h: \Sigma \rightarrow \mathbb{R}$ then for every w everywhere tangent to Σ such that $(di)^{-1}w \in \mathcal{X}(\Sigma, \sigma)$,

$$\begin{aligned}
(\omega^D_{(\Sigma, \sigma)})_i(v, w) &= \int_{\Sigma} \omega(v, w)\sigma = \int_{\Sigma} dh((di)^{-1}w)\sigma \\
&= \int_{\Sigma} (\mathcal{L}_{(di)^{-1}w}h)\sigma = \int_{\Sigma} \mathcal{L}_{(di)^{-1}w}(h\sigma) = \mathcal{L}_{(di)^{-1}w} \int_{\Sigma} h\sigma \\
&= \lim_{t \rightarrow 0} \frac{\phi_t^* \int_{\Sigma} h\sigma - \int_{\Sigma} h\sigma}{t} = 0.
\end{aligned} \tag{2.5}$$

The fourth equality is since $\mathcal{L}_{(di)^{-1}w}\sigma = 0$ and the fact that $\mathcal{L}_{(di)^{-1}w}(h\sigma) = (\mathcal{L}_{(di)^{-1}w}h)\sigma + h(\mathcal{L}_{(di)^{-1}w}\sigma)$; the fifth equality is since Σ is compact, and the last equality is since for an orientation preserving integral curve $t \rightarrow \phi_t$ and a 2-form γ on Σ , $\int_{\Sigma} \gamma$ is invariant under pulling back by ϕ_t .

3 \Rightarrow 4 Assume that $\int_{\Sigma} \alpha_v(w)\sigma = 0$ for every w that is everywhere tangent to Σ and satisfies $\mathcal{L}_{(di)^{-1}w}\sigma = 0$. Decompose $v = \xi_v + \tau_v$, to a vector field ξ_v everywhere ω -orthogonal to Σ and a vector field τ_v everywhere tangent to Σ , as in (2.2), so $\alpha_v(\cdot) = \omega(\xi_v, di(\cdot)) + \omega(\tau_v, di(\cdot))$. By assumption α_v is closed; by Lemma 2.4 and the step 1 \Rightarrow 2 above, α_{ξ_v} is closed, hence α_{τ_v} is closed. When we consider $\tau_v: \Sigma \rightarrow di(T\Sigma) \xrightarrow{(di)^{-1}} T\Sigma$ as a vector field on Σ , we conclude that $\sigma((di)^{-1}\tau_v, \cdot)$ is closed on Σ . By Cartan's formula and since σ is closed, we get that $(di)^{-1}\tau_v \in \mathcal{X}(\Sigma, \sigma)$. By the assumption on v and the choice of ξ_v , for every w that is everywhere tangent to Σ and satisfies $\mathcal{L}_{(di)^{-1}w}\sigma = 0$,

$$0 = \int_{\Sigma} \alpha_v(w)\sigma = \int_{\Sigma} \sigma((di)^{-1}\tau_v, w)\sigma.$$

In particular,

$$\int_{\Sigma} \sigma((di)^{-1}\tau_v, (di)^{-1}\tau_v)\sigma = 0.$$

Thus (since Σ is connected and σ is a volume form) $\tau_v = 0$, so $\alpha_v = \alpha_{\xi_v}$ is everywhere ω -orthogonal to Σ .

4 \Rightarrow 1 By Lemma 2.4. □

Remark 2.6 Notice that the steps 1 \Rightarrow 2, 2 \Rightarrow 3 and 4 \Rightarrow 1 hold for every $v \in T_i C^\infty(\Sigma, M)$. Only the step 3 \Rightarrow 4 requires $v \in \Gamma_{\text{closed}}(i^*TM)$. ○

Lemma 2.7. *Let $\Sigma = (\Sigma, \sigma) \xrightarrow{i} (M, \omega)$ be an embedded closed connected symplectic submanifold of dimension 2. The map $v \mapsto di^{-1}\tau_v$ from $\Gamma_{\text{closed}}(i^*TM)$ is onto $\mathcal{X}(\Sigma, \sigma)$ and restricts to a one-to-one and onto map from the subspace $\{\tau_v \mid v \in \Gamma_{\text{closed}}(i^*TM)\}$.*

Proof. First, we show that the image is a subset of $\mathcal{X}(\Sigma, \sigma)$: by assumption, $v \in \Gamma_{\text{closed}}(i^*TM)$; by Lemma 2.4 and $1 \Rightarrow 2$ in Lemma 2.5, $\xi_v \in \Gamma_{\text{exact}}(i^*TM) \subset \Gamma_{\text{closed}}(i^*TM)$, hence, $\tau_v = v - \xi_v$ is in the space $\Gamma_{\text{closed}}(i^*TM)$. In other words,

$$d\iota_{\tau_v}\omega|_{di(T\Sigma)} = 0. \quad (2.6)$$

Therefore, since i is a symplectic embedding, $d\iota_{di^{-1}\tau_v}\sigma = 0$. Thus, by Cartan's formula, since σ is a closed form, $\mathcal{L}_{di^{-1}\tau_v}\sigma = 0$. Reversing the argument, we get that for every $\tau \in \mathcal{X}(\Sigma, \sigma)$, the vector $di\tau$ is a vector in $\Gamma_{\text{closed}}(i^*TM)$ that is everywhere tangent to Σ , hence the map is onto.

By (2.6), the space $\{\tau_v \mid v \in \Gamma_{\text{closed}}(i^*TM)\}$ is a subspace of $\Gamma_{\text{closed}}(i^*TM)$. By the above argument, the map $\tau_v \mapsto di^{-1}\tau_v$ on it is onto $\mathcal{X}(\Sigma, \sigma)$. □

Corollary 2.8. *Let $\Sigma = (\Sigma, \sigma) \xrightarrow{i} (M, \omega)$ be an embedded closed connected symplectic submanifold of dimension 2. Then*

$$\Gamma_{\text{closed}}(i^*TM) = \Gamma_{\text{exact}}(i^*TM) \oplus \mathcal{X}(\Sigma, \sigma).$$

*The splitting gives a convenient space structure on $\Gamma_{\text{exact}}(i^*TM) \oplus \mathcal{X}(\Sigma, \sigma)$.*

Proof. A vector $v \in \Gamma_{\text{closed}}(i^*TM)$ decomposes as $\xi_v + \tau_v$ where ξ_v is everywhere ω -orthogonal to Σ and τ_v is everywhere tangent to Σ . Such a decomposition exists and is unique, e.g., by Remark A.9 and Corollary A.8.

By Lemma 2.5, the space $\{\xi_v \mid v \in \Gamma_{\text{closed}}(i^*TM)\}$ equals $\Gamma_{\text{exact}}(i^*TM)$. By Lemma 2.7, the space $\{\tau_v \mid v \in \Gamma_{\text{closed}}(i^*TM)\}$ is identified with $\mathcal{X}(\Sigma, \sigma)$. Notice that the maps $v \xrightarrow{h_1} (\xi_v, (di)^{-1}\tau_v)$ and $(\xi, w) \xrightarrow{h_2} \xi + diw$ send smooth curves to smooth curves. Moreover, for $c_1: \mathbb{R} \rightarrow \Gamma_{\text{exact}}(i^*TM) \oplus \mathcal{X}(\Sigma, \sigma)$, if $c_2: \mathbb{R} \rightarrow \Gamma_{\text{closed}}(i^*TM)$ satisfies $c_2' = h_2(c_1)$ then $(h_1(c_2))' = c_1$.

The space $\Gamma_{\text{closed}}(i^*TM)$ is convenient since it is the kernel of the continuous map $v \mapsto \alpha_v$ composed on $\alpha \rightarrow d\alpha$, from the convenient space $\Gamma(i^*TM)$ to the space $\Omega^1(\Sigma)$ of 1-forms on Σ and then to $\Omega^2(\Sigma)$. □

Proof of Proposition 2.2. Given $i \in \mathcal{S}_e(\Sigma, \sigma)$, by Weinstein's symplectic tubular neighbourhood theorem (see §2.3), the symplectic embedding i can be extended on a neighbourhood U of the zero section in $N\Sigma$ to a symplectic embedding $\Phi_i: U \rightarrow M$. By the identification of each fiber $N_x\Sigma$ with $(T_{i(x)}i(\Sigma))^\omega$, and Lemma 2.5, the elements of U are of the form $(y, \xi(y))$ where ξ is in $\Gamma_{\text{exact}}(i^*TM)$. The space $\mathcal{X}(\Sigma, \sigma)$ is the Lie algebra of $\text{Symp}(\Sigma, \sigma)$, see [5, 43.12]. Let V_e be a chart neighbourhood of the identity map $e \in \text{Symp}(\Sigma, \sigma)$ and denote by

$$\psi_e: V_e \rightarrow \mathcal{X}(\Sigma, \sigma)$$

the corresponding chart in an atlas on $\text{Symp}(\Sigma, \sigma)$. Define

$$W_i := \{\ell \in \mathcal{S}_e(\Sigma, \sigma) \mid \ell(x) = \Phi_i(b(x), \xi(b(x))) \text{ for } \xi \in \Gamma_{\text{exact}}(i^*TM), b \in V_e \text{ s.t. } (b(x), \xi(b(x))) \in U \forall x \in \Sigma\},$$

$$\phi_i: W_i \rightarrow \Gamma_{\text{exact}}(i^*TM) \oplus \mathcal{X}(\Sigma, \sigma), \quad \phi_i(\ell) := (\xi, \psi_e(b)).$$

By part (2) of Corollary 2.8, $\Gamma_{\text{exact}}(i^*TM) \oplus \mathcal{X}(\Sigma, \sigma)$ is a convenient space. The set $\{(b(x), \xi(b(x))) \in U \forall x \in \Sigma\}$ is c^∞ -open in $\Gamma_{\text{exact}}(i^*TM) \oplus \mathcal{X}(\Sigma, \sigma)$. Thus ϕ_i is a bijection of W_i onto a c^∞ -open subset of $\Gamma_{\text{exact}}(i^*TM) \oplus \mathcal{X}(\Sigma, \sigma)$. The collection $(W_i, \phi_i)_{i \in \mathcal{S}_e(\Sigma, \sigma)}$ defines a smooth atlas on $\mathcal{S}_e(\Sigma, \sigma)$: the chart changings ϕ_{ik} are smooth by smoothness of the exponential map and of each symplectic embedding Φ_i . □

Lemma 2.9. *Let $\Sigma = (\Sigma, \sigma) \xrightarrow{i} (M, \omega)$ be an embedded closed connected symplectic submanifold of dimension 2.*

1. *For a section $v: \Sigma \rightarrow i^*TM$ that is in $T_i \mathcal{S}_e(\Sigma, \sigma)$, the form*

$$\alpha_v = \omega(v, di(\cdot))$$

is a closed form on Σ .

2. *Every vector $v \in \Gamma_{\text{closed}}(i^*TM)$ can be extended to a vector field \tilde{v} on a neighbourhood of $i(\Sigma)$ in M such that $\mathcal{L}_{\tilde{v}}\omega = 0$.*

Proof. 1. For a vector field v , let $t \rightarrow \phi_t$ be the integral curve of v starting at i . Since v is tangent to $\mathcal{S}_e(\Sigma, \sigma)$, for w_1, w_2 in $T\Sigma$, we get $\phi_t^*\omega(di w_1, di w_2) = \sigma(w_1, w_2) = \omega(di w_1, di w_2)$, where $di: T\Sigma \rightarrow T(i(\Sigma))$. Hence on $i(\Sigma)$,

$$\mathcal{L}_v\omega = \lim_{t \rightarrow 0} \frac{\phi_t^*\omega - \omega}{t} = 0.$$

By Cartan's formula, this implies $d\iota_v\omega = 0$ as a form on $i(\Sigma)$, i.e., α_v is a closed form on Σ .

2. By Cartan's formula and the fact that ω is a closed form, we need to extend v to \tilde{v} on a neighbourhood of $i(\Sigma)$ in M such that the 1-form $\omega(\tilde{v}, \cdot)$ is a closed form. By the decomposition (2.2) and Lemma 2.4, it is enough to extend $\tau_v \in \Gamma_{\text{closed}}(i^*TM)$ that is everywhere tangent to Σ to such a vector. The closed form $\iota_{(di)^{-1}\tau_v}\sigma$ on the zero section of $(N\Sigma, \omega_{N\Sigma})$ pulls back (through the projection of $N\Sigma$ to the zero section) to a closed one-form on $N\Sigma$ that is consistent with $\iota_{(di)^{-1}\tau_v}\sigma$ on the zero section and zero on directions $\omega_{N\Sigma}$ -orthogonal to the zero section. By Weinstein's symplectic tubular neighbourhood theorem, the push forward of this form via the symplectic embedding $\Phi_i: (U, \omega_{N\Sigma}) \rightarrow (M, \omega)$ of (2.3) is a closed one-form $\tilde{\alpha}_{\tau_v}$ on a neighbourhood of $i(\Sigma)$ in M that is consistent with $\iota_{\tau_v}\omega$ on vectors tangent to $i(\Sigma)$. Define $\tilde{\tau}_v$ to be the vector field such that $\tilde{\alpha}_{\tau_v}(\cdot) = \omega(\tilde{\tau}_v, \cdot)$. The vector $\tilde{\tau}_v$ is well defined since ω is non-degenerate. □

Denote by $\omega_{\mathcal{S}_e(\Sigma, \sigma)}$ the pullback (through inclusion) of $\omega_{\mathcal{S}_i(\Sigma)}$ to $\mathcal{S}_e(\Sigma, \sigma)$, and by $\omega^D_{\mathcal{S}_e(\Sigma, \sigma)}$ the pullback of $\omega^D_{(\Sigma, \sigma)}$ to $\mathcal{S}_e(\Sigma, \sigma)$.

Proposition 2.10. *The 2-form $\omega^D_{\mathcal{S}_e(\Sigma, \sigma)}$ on $\mathcal{S}_e(\Sigma, \sigma)$ is closed and weakly non-degenerate.*

Proof. The form is closed as the restriction of the closed form $\omega^D_{(\Sigma, \sigma)}$ (see proposition 2.1) to the manifold $\mathcal{S}_e(\Sigma, \sigma)$ (see proposition 2.2). We need to show that it is weakly non-degenerate.

1. For $0 \neq \tau_v \in T_i \mathcal{S}_e(\Sigma, \sigma)$ that is everywhere tangent to Σ ,

$$\omega^D(\tau_v, \tau_v) = \int_{\Sigma} \omega(\tau_v, \tau_v)\sigma = \int_{\Sigma} \sigma(di^{-1}\tau_v, di^{-1}\tau_v)\sigma \neq 0.$$

The last inequality is since Σ is connected, σ is a volume form, $\tau_v \neq 0$ and $di: T\Sigma \rightarrow Ti(\Sigma)$ is an isomorphism.

2. Suppose that $w \in T_i(\mathcal{S}_e(\Sigma, \sigma))$ is not tangent to Σ at $x \in \Sigma$. By Lemma 2.4, $w = \xi_w + \tau_w$, where $\xi_w = i^*\xi$ with ξ a Hamiltonian vector field on M and τ_w everywhere tangent to Σ . In particular, $\xi_w(x) \neq 0$. Let w_1 be a vector in $(i^*(TM))_x$ such that

$$\omega(\xi_w(x), w_1) > 0,$$

and w_1 is symplectically orthogonal to $di_x(T_x \Sigma)$. (For example, $w_1 = J\xi_w(x)$ for an almost complex structure J that is ω -compatible. See part (2) of Claim A.7 and Remark A.9.)

Now extend w_1 to a section $w_1: \Sigma \rightarrow i^*(TM)$ in $T_i \mathcal{S}_e(\Sigma, \sigma)$ such that $\omega(\xi_w(y), w_1(y)) > 0$ and $w_1(y)$ is symplectically orthogonal to $di_y(T_y \Sigma)$ for y in a small neighborhood of x , and vanishing outside it. By Lemma 2.4, $w_1 = i^*W_H$ with W_H a Hamiltonian vector field on M . By Lemma 2.9, $w \in \Gamma_{\text{closed}}(i^*TM)$, hence (see Lemma 2.7), τ_w is an everywhere tangent to Σ vector that satisfies $\mathcal{L}_{di^{-1}\tau_w}\sigma = 0$. Then

$$(\omega^D_{\mathcal{S}_e(\Sigma, \sigma)})_i(w, w_1) = (\omega^D_{\mathcal{S}_e(\Sigma, \sigma)})_i(\xi_w, w_1) = \int_{\Sigma} \omega(\xi_w, w_1)\sigma \neq 0,$$

where the first equality follows from $1 \Rightarrow 3$ in Lemma 2.5 and the last inequality follows from the choice of w_1 .

□

3 The quotient of $\mathcal{S}_e(\Sigma, \sigma)$ by symplectic reparametrizations

The forms $\omega_{\mathcal{S}_e(\Sigma, \sigma)}$ and $\omega^D_{\mathcal{S}_e(\Sigma, \sigma)}$ coincide in exact directions

Claim 3.1. *For tangent vectors $v_1, v_2 \in T_f \mathcal{S}_e(\Sigma, \sigma)$, the integrand $\iota_{((v_1, 0) \wedge (v_2, 0))} \text{ev}^*(\omega \wedge \omega)$ equals*

$$2\omega(v_1, v_2)\omega(df(\cdot), df(\cdot)) + 2\omega(v_1, df(\cdot)) \wedge \omega(v_2, df(\cdot)) = 2\omega(v_1, v_2)f^*\omega(\cdot, \cdot) + 2\omega(v_1, df(\cdot)) \wedge \omega(v_2, df(\cdot)),$$

which, since $f \in \mathcal{S}_e(\Sigma, \sigma)$, equals

$$2\omega(v_1, v_2)\sigma(\cdot, \cdot) + 2\omega(v_1, df(\cdot)) \wedge \omega(v_2, df(\cdot)). \quad (3.7)$$

Lemma 3.2. *For $u \in T_f \mathcal{S}_e(\Sigma, \sigma)$ such that $\omega(u, df(\cdot))$ on Σ is exact, we get that*

$$\iota_u(\omega_{\mathcal{S}_e(\Sigma, \sigma)})_f = 2 \int_{\Sigma} \omega(u, \cdot)\sigma = 2\iota_u(\omega^D_{\mathcal{S}_e(\Sigma, \sigma)})_f,$$

i.e., the two forms coincide in exact directions, up to multiplication by a constant.

Proof. Indeed, since $\omega(u, df(\cdot)) = dh$ for a function $h: \Sigma \rightarrow \mathbb{R}$, and for every $v \in T_f \mathcal{S}_e(\Sigma, \sigma)$ the form α_v on Σ is closed (see Lemma 2.9), the integration of the second term in (3.7) along $f(\Sigma)$ vanishes:

$$\int_{\Sigma} \omega(v, df(\cdot)) \wedge \omega(u, df(\cdot)) = \int_{\Sigma} \alpha_v \wedge dh = \int_{\Sigma} \alpha_v \wedge dh - d\alpha_v \wedge h = \int_{\Sigma} d(\alpha_v \wedge h) = \int_{\partial(\Sigma)} \alpha_v \wedge h = 0.$$

□

Note that (in the second equality) we used the fact that α_v on Σ is closed (by Lemma 2.9), so the argument works in the space of symplectic embeddings and not for arbitrary embeddings.

Directions of degeneracy for the form $\omega_{\mathcal{S}_e(\Sigma, \sigma)}$

Lemma 3.3. *Let $i \in \mathcal{S}_e(\Sigma, \sigma)$.*

1. *If $w \in T_i \mathcal{S}_e(\Sigma, \sigma)$ is not everywhere tangent to Σ , then there exists $w_1 \in T_i \mathcal{S}_e(\Sigma, \sigma)$ satisfying $w_1 = i^* W_H$ with W_H a Hamiltonian vector field on M such that $(\omega^D_{\mathcal{S}_e(\Sigma, \sigma)})_i(w, w_1) \neq 0$.*
2. *If $\tau \in T_i \mathcal{S}_e(\Sigma, \sigma)$ is everywhere tangent to Σ , then $\iota_\tau(\omega_{\mathcal{S}_e(\Sigma, \sigma)})_i = 0$.*
3. *If $w \in T_i \mathcal{S}_e(\Sigma, \sigma)$ is not everywhere tangent to Σ , then there exists $w_1 \in T_i \mathcal{S}_e(\Sigma, \sigma)$ satisfying $w_1 = i^* W_H$ with W_H a Hamiltonian vector field on M such that $(\omega_{\mathcal{S}_e(\Sigma, \sigma)})_i(w, w_1) \neq 0$.*

Proof. 1. This is shown in the proof of Proposition 2.10.

2. Suppose that τ is everywhere tangent to Σ . Lift τ to a vector field $\ell = (\tau, 0)$; let $\tau_2 \in T_i \mathcal{S}_e(\Sigma, \sigma)$ and ℓ_2 a lifting of τ_2 . We show that the integrand $\iota_{\ell \wedge \ell_2} \text{ev}^*(\omega \wedge \omega)$ vanishes when restricted to $T(\{i\} \times \Sigma)$. Indeed, for $z_1, z_2 \in T_x(\{i\} \times \Sigma)$, by definition and Lemma 3.5 below,

$$\iota_{\ell \wedge \ell_2} \text{ev}^*(\omega \wedge \omega)_x(z_1, z_2) = \iota_{\tau \wedge \text{d} \text{ev}(\ell_2)}(\omega \wedge \omega)_{i(x)}(\text{d}f(z_1), \text{d}i(z_2)).$$

So it is enough to show that

$$\iota_{\tau \wedge \text{d} \text{ev}(\ell_2)}(\omega \wedge \omega)|_{\text{d}i(T_x \Sigma)}$$

vanishes. This follows from Lemma 3.6, since, by assumption, $\tau(x) \in \text{d}i_x(T_x \Sigma)$ and $\text{d}i_x(T_x \Sigma) \subset T_{i(x)} M$ is a two-dimensional subspace.

3. By item (2) and Lemma 2.4 it is enough to prove item (3) with the assumption that $w = i^* V_H$ with V_H a Hamiltonian vector field on M . By Lemma 3.2, this case follows from item (1). □

Remark 3.4 By the same argument we get that also on $\mathcal{S}_i(\Sigma)$, we have $\omega_{\mathcal{S}_i(\Sigma)}(\tau, \cdot) = 0$ at f iff τ is tangent to $f(\Sigma)$ at every $x \in \Sigma$. See [2, Thm 1]. ⊙

Lemma 3.5. *For $(\nu, v_\Sigma) \in T(C^\infty(\Sigma, M) \times \Sigma)$,*

$$\text{d}(\text{ev})_{(f, x)}(\nu, v_\Sigma) = \nu_f(x) + \text{d}f_x(v_\Sigma)$$

In particular,

$$\text{d}(\text{ev})|_{T(\{f\} \times \Sigma)} = \text{d}f, \tag{3.8}$$

and

$$\text{d}(\text{ev})_{(f, x)}(\nu, 0) = \nu_f(x). \tag{3.9}$$

Lemma 3.6. *Let W be a vector space, and let α be a 4-form: $\alpha: \bigwedge^4 W \rightarrow \mathbb{R}$. Let $V \subset W$ be a subspace of dimension ≤ 2 . Then $(\iota_{(v \wedge w)} \alpha)|_V = 0$ for all $v \in V, w \in W$.*

Proof. This is since any three vectors in V are linearly dependent. □

As a result of Lemma 3.3 and Lemma 2.4, we get an extension of Lemma 3.2:

Corollary 3.7. *For every $u \in T_f \mathcal{S}_e(\Sigma, \sigma)$, there is $w \in T_f \mathcal{S}_e(\Sigma, \sigma)$ that is everywhere tangent to f , such that $\omega(u + w, \text{d}f(\cdot))$ on Σ is exact and*

$$\iota_u(\omega_{\mathcal{S}_e(\Sigma, \sigma)})_f = \iota_{u+w}(\omega_{\mathcal{S}_e(\Sigma, \sigma)})_f = 2 \int_\Sigma \omega(u + w, \cdot) \sigma = 2 \iota_{u+w}(\omega^D_{\mathcal{S}_e(\Sigma, \sigma)})_f,$$

w equals zero if $\omega(u, \text{d}f(\cdot))$ on Σ is exact.

A symplectic form on the quotient by symplectic reparametrizations

The group $\text{Symp}(\Sigma, \sigma)$ of symplectomorphisms of (Σ, σ) is a Lie group in the Convenient Setup, its Lie algebra is $\mathcal{X}(\Sigma, \sigma)$ [5, 43.12]. The Lie group $\text{Symp}(\Sigma, \sigma)$ acts freely on $\mathcal{S}_e(\Sigma, \sigma)$ on the right by

$$\psi.i = i \circ \psi^{-1}. \quad (3.10)$$

Denote the quotient map

$$q: \mathcal{S}_e(\Sigma, \sigma) \rightarrow \mathcal{S}_e(\Sigma, \sigma) / \text{Symp}(\Sigma, \sigma). \quad (3.11)$$

Denote by $\text{Ham}(M, \omega)$ the Lie group of Hamiltonian symplectomorphisms of (M, ω) ; its Lie algebra \mathfrak{ham} is the space of Hamiltonian vector fields [5, 43.12, 43.13]. (A vector field X on M is Hamiltonian if the form $\iota_X \omega$ is exact; a symplectomorphism of (M, ω) is Hamiltonian if it is the time one flow of a time dependent Hamiltonian vector field.) The group $\text{Ham}(M, \omega) \subset \text{Symp}(M, \omega)$ acts freely on $\mathcal{S}_e(\Sigma, \sigma)$ on the left by

$$\phi.i = \phi \circ i.$$

The action descends to the quotient $\mathcal{S}_e(\Sigma, \sigma) / \text{Symp}(\Sigma, \sigma)$ as

$$\phi.[i] = [\phi \circ i].$$

Denote by $\text{Emb}(\Sigma, M)$ the space of embeddings of Σ into M . (An open set in the manifold $C^\infty(\Sigma, M)$.) The Lie group $\text{Diff}(\Sigma)$ [5, 43.1] of reparametrizations of Σ acts freely on $\text{Emb}(\Sigma, M)$ on the right by

$$\psi.i = i \circ \psi^{-1}.$$

The group $\text{Ham}(M, \omega)$ acts on $\text{Emb}(\Sigma, M)$, (and its quotient by $\text{Diff}(\Sigma)$), by $\phi.i = \phi \circ i$.

Lemma 3.8. *A path-connected component of the space $\mathcal{S}_e(\Sigma, \sigma)$ modulo the action of $\text{Symp}(\Sigma, \sigma)$ is identified with a path-connected component of the space $\text{Emb}(\Sigma, M)$ modulo the action of $\text{Diff}(\Sigma)$ as follows. Let $f: \Sigma \rightarrow M$ be an embedding that is connected to an element $i \in \mathcal{S}_e(\Sigma, \sigma)$ through a path in $\text{Emb}(\Sigma, M)$. There exists a Σ -reparametrization φ_f such that $f \circ \varphi_f \in \mathcal{S}_e(\Sigma, \sigma)$; the Σ -reparametrization φ_f is unique up to symplectic Σ -reparametrizations. The map $[f] \mapsto [f \circ \varphi_f]$ from the quotient of the path-connected component of $i \in \mathcal{S}_e(\Sigma, \sigma)$ in $\text{Emb}(\Sigma, M)$ by $\text{Diff}(\Sigma)$ to $\mathcal{S}_e(\Sigma, \sigma) / \text{Symp}(\Sigma, \sigma)$ is well defined, one-to-one and its image is the quotient of the path-connected component of i in $\mathcal{S}_e(\Sigma, \sigma)$ by the action of $\text{Symp}(\Sigma, \sigma)$.*

The map $f \mapsto f \circ \varphi_f$ sends a $\text{Ham}(M, \omega)$ -orbit in $\text{Emb}(\Sigma, M)$ to a $\text{Ham}(M, \omega)$ -orbit in $\mathcal{S}_e(\Sigma, \sigma)$ modulo symplectic reparametrizations.

Proof. Let $\{f_t\}_{0 \leq t \leq 1}$ be a path in $\text{Emb}(\Sigma, M)$ with $f_0 = i \in \mathcal{S}_e(\Sigma, \sigma)$ and $f_1 = f$. Set

$$\omega_t = f_t^* \omega.$$

By definition ω_t is closed. By the Homotopy invariance of de Rham cohomology, $[\omega_t] = [\omega_0]$ for all t . Therefore

$$\int_{\Sigma} f_t^* \omega - \int_{\Sigma} f_0^* \omega = \int_{\Sigma} d\eta = \int_{\partial \Sigma} \eta = 0, \quad (3.12)$$

where the last equality is since Σ is closed and the equality before it is by Stoke's theorem. Since $f_0^* \omega = i^* \omega = \sigma$, equation (3.12) implies that for all t , the integral $\int_{\Sigma} \omega_t = \int_{\Sigma} f_t^* \omega \neq 0$ therefore, the 2-form ω_t is

a non-vanishing volume form on the 2-dimensional manifold Σ hence non-degenerate. We can then apply Moser's theorem [10] to get an isotopy $\varphi: \Sigma \times \mathbb{R} \rightarrow \Sigma$ such that

$$(f_t \circ \varphi_t)^* \omega = \varphi_t^* \omega_t = \omega_0 = \sigma \text{ for } 0 \leq t \leq 1. \quad (3.13)$$

Set $\varphi_f := \varphi_1$.

Notice that if $\psi_1, \psi_2 \in \text{Diff}(\Sigma)$ are such that $(f \circ \psi_i)^* \omega = \sigma$, then

$$(\psi_2^{-1} \circ \psi_1)^* \sigma = (\psi_2^{-1} \circ \psi_1)^* (f \circ \psi_2)^* \omega = (f \circ \psi_2 \circ \psi_2^{-1} \circ \psi_1)^* \omega = (f \circ \psi_1)^* \omega = \sigma,$$

so $[f \circ \psi_1] = [f \circ \psi_2]$ in $\mathcal{S}_e(\Sigma, \sigma)$ modulo $\text{Symp}(\Sigma, \sigma)$. Therefore the class in $\mathcal{S}_e(\Sigma, \sigma)$ modulo $\text{Symp}(\Sigma, \sigma)$ we associated to f is independent of the choice of path and isotopy. Similarly, it is independent of the choice of the representative of $[f]$: if $g: \Sigma \rightarrow M$ and $\alpha, \beta \in \text{Diff}(\Sigma)$ are such that $f = g \circ \alpha$ and $(g \circ \beta)^* \omega = \sigma$, then

$$(\beta^{-1} \circ \alpha \circ \varphi_f)^* \sigma = (\beta^{-1} \circ \alpha \circ \varphi_f)^* (g \circ \beta)^* \omega = (g \circ \beta \circ \beta^{-1} \circ \alpha \circ \varphi_f)^* \omega = (f \circ \varphi_f)^* \omega = \sigma,$$

so $[f \circ \varphi_f] = [g \circ \beta]$. Therefore the assignment $[f] \mapsto [f \circ \varphi_f]$ is well defined. It is also one-to-one: if g and f are not in the same class in $\text{Emb}(\Sigma, M)$ modulo $\text{Diff}(\Sigma)$, the associated $g \circ \varphi_g$ and $f \circ \varphi_f$ cannot be equal up to symplectic reparametrization of Σ . By the construction of φ_f (see (3.13)), the map $f \mapsto f \circ \varphi_f$ sends paths starting from i to paths starting from i composed on a symplectic reparametrization of (Σ, σ) , hence the image of $[f] \mapsto [f \circ \varphi_f]$ is contained in the quotient of the path-connected component of i in $\mathcal{S}_e(\Sigma, \sigma)$ modulo $\text{Symp}(\Sigma, \sigma)$; it is onto this quotient since for every symplectic embedding h that is path-connected to i in $\mathcal{S}_e(\Sigma, \sigma)$, the class of h in C modulo $\text{Diff}(\Sigma)$ is sent to the class of h in $\mathcal{S}_e(\Sigma, \sigma)$ modulo $\text{Symp}(\Sigma, \sigma)$.

Since the $\text{Ham}(M, \omega)$ -actions are from the left while the actions of $\text{Diff}(\Sigma)$ and $\text{Symp}(\Sigma, \sigma)$ are from the right, the map $f \mapsto f \circ \varphi_f$ sends an $\text{Ham}(M, \omega)$ -orbit in $\text{Emb}(\Sigma, M)$ to an $\text{Ham}(M, \omega)$ -orbit in $\mathcal{S}_e(\Sigma, \sigma)$ modulo symplectic reparametrizations. For $\phi \in \text{Ham}(M, \omega)$, the embedding $h = \phi \circ f$ is connected to $\phi \circ i \in \mathcal{S}_e(\Sigma, \sigma)$ through a path in $\text{Emb}(\Sigma, M)$, hence there exists $\varphi_h \in \text{Diff}(\Sigma)$ such that $h \circ \varphi_h \in \mathcal{S}_e(\Sigma, \sigma)$. Since

$$(\varphi_f^{-1} \circ \varphi_h)^* \sigma = (\varphi_f^{-1} \circ \varphi_h)^* (f \circ \varphi_f)^* \omega = (f \circ \varphi_h)^* \omega = (\phi^{-1} \circ h \circ \varphi_h)^* \omega = (h \circ \varphi_h)^* \phi^{-1*} \omega = (h \circ \varphi_h)^* \omega = \sigma,$$

the symplectic embeddings $h \circ \varphi_h = (\phi \circ f) \circ \varphi_h$ and $\phi \circ (f \circ \varphi_f)$ are in the same class of $\mathcal{S}_e(\Sigma, \sigma)$ modulo $\text{Symp}(\Sigma, \sigma)$. □

3.9 A neighbourhood of an element $[i]$ in $\text{Emb}(\Sigma, M)$ modulo $\text{Diff}(\Sigma)$ is identified with a neighbourhood of the zero section in the space of sections of the Normal bundle $N\Sigma$. The identification is as follows: choose a Riemannian metric on M such that $N\Sigma$ is the orthogonal complement of $T\Sigma = T(i\Sigma)$ in $TM|_{i(\Sigma)}$; the exponential map with respect to that metric sends a neighbourhood of the zero section to a neighbourhood of the submanifold $i(\Sigma)$. This defines an atlas on $\text{Emb}(\Sigma, M)/\text{Diff}(\Sigma)$, all translation functions are smooth by the smoothness of the exponential map. Hence, by Lemma 3.8, we get a smooth structure on $\mathcal{S}_e(\Sigma, \sigma)$ modulo $\text{Symp}(\Sigma, \sigma)$.

The map \exp sends an orbit of the linearized action of $\text{Ham}(M, \omega)$ on $N\Sigma$ onto an orbit of $\text{Ham}(M, \omega)$ in $\text{Emb}(\Sigma, M)$ modulo reparametrizations. ⊗

Corollary 3.10. *A path of symplectic embeddings $(\Sigma, \sigma) \rightarrow (M, \omega)$ through f can be written, up to symplectic reparametrizations of (Σ, σ) , as $\Psi_t \circ f$, where Ψ_t is a path in $\text{Ham}(M, \omega)$.*

Proof. Note that for a time dependent vector field $v_t \in T_i \mathcal{S}_e(\Sigma, \sigma) \times \mathbb{R}$, the decomposition (2.2) is smooth with respect to the time parameter. Thus the corollary follows from Lemma 2.4 and Lemma 2.7. \square

Definition 3.11 [6, Definition 10]. Let (X, Ω) be a weakly symplectic manifold. Let $G \curvearrowright X$ be a free action of a Lie group G on X , such that $g^* \Omega = \Omega$ for all $g \in G$ ($g: X \rightarrow X$ denotes the action $x \rightarrow g.x$). The collection of subspaces

$$\mathcal{D}_x := \{v \in T_x X \mid \Omega_x(v, \xi_X(x)) = 0 \ \forall \xi \in \mathfrak{g}\}$$

for $x \in X$ defines a distribution \mathcal{D} on X . Let $i_N: N \hookrightarrow X$ be a maximal integral manifold of \mathcal{D} and let $q: X \rightarrow X/G$ denote the projection to the orbit space. Suppose that the quotient induces a smooth structure on $q(N)$, in which there exists a unique weak symplectic structure Ω_{red} on $q(N)$ such that $(q|_N)^* \Omega_{\text{red}} = i_N^* \Omega$. Then the weakly symplectic manifold $(q(N), \Omega_{\text{red}})$ will be called a *reduction* or *symplectic quotient* of (X, Ω) with respect to the G -action. \oslash

A *distribution* on a smooth manifold M assigns to each point $x \in M$ a c^∞ -closed subspace \mathcal{D}_x of $T_x M$. (The c^∞ -topology on a locally convex space E is the finest topology for which all smooth curves $c: \mathbb{R} \rightarrow E$ are continuous.) If $\mathcal{D} = \{\mathcal{D}_x\}$ is a distribution on a manifold M and $i: N \hookrightarrow M$ is the inclusion map of a path-connected submanifold N of M , then N is called an *integral manifold* of \mathcal{D} if $d_i(T_x N) = \mathcal{D}_{i(x)}$ for all $x \in N$. An integral manifold of \mathcal{D} is called *maximal* if it is not properly contained in any other integral manifold. (Note that since the local model is a locally convex vector space, a manifold is path-connected if and only if it is connected.)

3.12 The motivation for Definition 3.11 comes from the standard reduction of a finite dimensional symplectic manifold (X, Ω) with respect to a Hamiltonian G -action with a moment map ϕ . For a regular value r of ϕ , the tangent space at p to the level surface $\phi^{-1}(r)$ is equal to the set \mathcal{D}_p of all vectors $v \in T_p X$ satisfying $\Omega(v, \xi_X(p)) = 0$ for all $\xi \in \mathfrak{g}$. The subspaces \mathcal{D}_p give a distribution \mathcal{D} on X , defined even in the absence of a moment map. If $G \curvearrowright X$ is a free symplectic action, then this distribution can be taken as the starting point of the optimal reduction method of Ortega and Ratiu [11]. \oslash

Theorem 3.13. *Consider $(\mathcal{S}_e(\Sigma, \sigma), \omega^D_{\mathcal{S}_e(\Sigma, \sigma)})$ with the action (3.10) of $\text{Symp}(\Sigma, \sigma)$. The $\text{Ham}(M, \omega)$ -orbit \mathcal{N} through $i \in \mathcal{S}_e(\Sigma, \sigma)$ is a maximal integral manifold of the distribution \mathcal{D} . The restriction of $\omega^D_{\mathcal{S}_e(\Sigma, \sigma)}$ to \mathcal{N} descends to a weak symplectic structure ω^D_{red} on the image $\mathcal{O} := q(\mathcal{N})$ in the orbit space under the projection $q: \mathcal{S}_e(\Sigma, \sigma) \rightarrow \mathcal{S}_e(\Sigma, \sigma)/\text{Symp}(\Sigma, \sigma)$. Thus the symplectic space $(\mathcal{O}, \omega^D_{\text{red}})$ is a reduction of $\mathcal{S}_e(\Sigma, \sigma)$ with respect to the $\text{Symp}(\Sigma, \sigma)$ -action.*

Proof. By Proposition 2.2 and Proposition 2.10, $(\mathcal{S}_e(\Sigma, \sigma), \omega^D_{\mathcal{S}_e(\Sigma, \sigma)})$ is a weakly symplectic manifold. Let $i \in \mathcal{S}_e(\Sigma, \sigma)$. By definition,

$$D_i = \{v \in T_i \mathcal{S}_e(\Sigma, \sigma) \mid \omega^D_i(v, \xi_{\mathcal{S}_e(\Sigma, \sigma)}(i)) = 0 \ \forall \xi \in \mathcal{X}(\Sigma, \sigma)\}.$$

By $1 \Leftrightarrow 3$ in Lemma 2.5, and the fact that $T_i \mathcal{S}_e(\Sigma, \sigma) \subset \Gamma_{\text{closed}}(i^* TM)$ (by Lemma 2.9),

$$T_i(\text{Ham}(M, \omega).i) = \{v \in T_i \mathcal{S}_e(\Sigma, \sigma) \mid \omega^D_i(v, \xi) = 0 \ \forall \xi \in \Gamma_{\text{closed}}(i^* TM) \text{ that is everywhere tangent to } \Sigma\}.$$

By Lemma 2.7, the space of vector fields in $\Gamma_{\text{closed}}(i^*TM)$ that are everywhere tangent to Σ is identified with the space $\mathcal{X}(\Sigma, \sigma)$. So $\text{Ham}(M, \omega)$ -orbits are integral manifolds of \mathcal{D} .

To see that maximal, assume that the $\text{Ham}(M, \omega)$ -orbit \mathcal{N} is properly contained in a (path-connected) integral manifold \tilde{N} . By Lemma 3.10 and Corollary 2.8, a path in \tilde{N} from an element $f \in \mathcal{N}$ to an element in $\tilde{N} \setminus \mathcal{N}$ can be written as $\Psi_t \circ f \circ \alpha_t$, where Ψ_t is a path in $\text{Ham}(M, \omega)$ and $\alpha_t: \Sigma \rightarrow \Sigma$ are in $\text{Symp}(\Sigma, \sigma)$; moreover, the generating vector field of the path v_t decomposes uniquely as $\xi_{v_t} + \tau_{v_t}$, where ξ_{v_t} is symplectically orthogonal to Σ and τ_{v_t} is everywhere tangent to Σ , and ξ_{v_t} is the vector field generating Ψ_t and τ_{v_t} is generating α_t . Since $d_{i_{\tilde{N}}}(\Gamma_x \tilde{N}) = \mathcal{D}_{i_{\tilde{N}}(x)}$ for all $x \in \tilde{N}$ (where $i_{\tilde{N}}$ is the inclusion map of \tilde{N} into M), and by $3 \Leftrightarrow 4$ in Lemma 2.5, for every $v_t \in d_{i_{\tilde{N}}}(\Gamma_x \tilde{N})$, the vector $v_t = \xi_{v_t}$ and $\tau_{v_t} = 0$. Hence $\alpha_t = \text{Id}$ for every t , and we get a path in $\text{Ham}(M, \omega)$ connecting an element in a $\text{Ham}(M, \omega)$ -orbit with an element outside the orbit: a contradiction.

Consider the inclusion-quotient diagram

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{i_{\mathcal{N}}} & \mathcal{S}_e(\Sigma, \sigma) \\ \downarrow q & & \\ \mathcal{O} & & \end{array}$$

By Lemma 3.8 and §3.9, the quotient induces a smooth structure on $\mathcal{O} = q(\mathcal{N})$. The pullback under the inclusion $i_{\mathcal{N}}$ of the form $\omega_{\mathcal{S}_e(\Sigma, \sigma)}^D$ to \mathcal{N} is closed, horizontal (by Lemma 2.5 and Lemma 2.7) and invariant to the action of $\text{Symp}(\Sigma, \sigma)$, hence basic. Therefore it is the pullback under the quotient q of a closed 2-form ω_{red}^D on \mathcal{O} . The reduced form is given by

$$\omega_{\text{red}}^D([v_1], [v_2]) = \int_{\Sigma} \omega(v_1, v_2) \sigma.$$

By part (1) of Lemma 3.3, the form ω_{red}^D is weakly non-degenerate. □

Remark 3.14 Similarly we get a closed 2-form $\omega_{\mathcal{S}_e(\Sigma, \sigma)}^{\text{red}}$ on \mathcal{O} :

$$\omega_{\mathcal{S}_e(\Sigma, \sigma)}^{\text{red}}([v_1], [v_2]) = \omega_{\mathcal{S}_e(\Sigma, \sigma)}(v_1, v_2),$$

well defined and weakly non-degenerate by Lemma 3.3. By Lemma 3.2, we get $\omega_{\mathcal{S}_e(\Sigma, \sigma)}^{\text{red}} = 2\omega_{\text{red}}^D$ on \mathcal{O} . However, in that case the $\text{Ham}(M, \omega)$ -orbits are not integral manifolds of the distribution \mathcal{D} given by

$$D_i = \{v \in T_i \mathcal{S}_e(\Sigma, \sigma) \mid \omega_{\mathcal{S}_e(\Sigma, \sigma)}(v, \xi_{\mathcal{S}_e(\Sigma, \sigma)}(i)) = 0 \forall \xi \in \mathcal{X}(\Sigma, \sigma)\}.$$

The reason for the difference is that for $v \in T_i \mathcal{S}_e(\Sigma, \sigma)$ the form $\iota_v \omega_{\mathcal{S}_e(\Sigma, \sigma)}^D|_{\mathcal{X}(\Sigma, \sigma)} \equiv 0$ iff $v = i^*V_H$ for a Hamiltonian vector field V_H on M (by Lemma 2.5 and Lemma 2.7) whereas $\iota_v \omega_{\mathcal{S}_e(\Sigma, \sigma)}|_{\mathcal{X}(\Sigma, \sigma)} \equiv 0$ for every $v \in T \mathcal{S}_e(\Sigma, \sigma)$, by part (2) of Lemma 3.3. From the same reason, the form $\omega_{\mathcal{S}_e(\Sigma, \sigma)}$ is degenerate. \oslash

4 Corollary to moduli spaces of J -holomorphic Σ -curves

For $J \in \mathcal{J}(M, \omega)$, denote by

$$\text{Ham}^J(M, \omega)$$

the subset of $\text{Ham}(M, \omega)$ of J -holomorphic Hamiltonian symplectomorphisms.

Lemma 4.1. *If f is an embedded (j, J) -holomorphic curve $f: \Sigma \rightarrow M$, and $g = \phi \circ f$ for $\phi \in \text{Ham}^J(M, \omega)$ then $g \in \mathcal{M}_e(A, \Sigma, J)$. If, in addition, $f^*\omega = \sigma$, then $g^*\omega = \sigma$.*

Proof. If $g = \phi \circ f$ for $\phi \in \text{Ham}^J(M, \omega)$ then g is an embedding $\Sigma \rightarrow M$, and

$$dg \circ j = d\phi \circ df \circ j = d\phi \circ J \circ df = J \circ d\phi \circ df = J \circ dg,$$

i.e., g is a (j, J) -holomorphic Σ -curve in M . If $f^*\omega = \sigma$, then

$$g^*\omega = (\phi \circ f)^*\omega = f^*\phi^*\omega = f^*\omega = \sigma.$$

□

Lemma 4.2. *Consider $f_1: \Sigma \rightarrow M$ and $f_2: \Sigma \rightarrow M$. Assume that $f_i^*\omega = \sigma$ for $i = 1, 2$. If ϕ is a diffeomorphism $\Sigma \rightarrow \Sigma$ such that $f_1 \circ \phi = f_2$, then $\phi \in \text{Symp}(\Sigma, \sigma)$.*

Proof.

$$\begin{aligned} \sigma(d\phi(\cdot), d\phi(\cdot)) &= f_1^*\omega(d\phi(\cdot), d\phi(\cdot)) = \omega(df_1 d\phi(\cdot), df_1 d\phi(\cdot)) \\ &= \omega(df_2(\cdot), df_2(\cdot)) = f_2^*\omega(\cdot, \cdot) \\ &= \sigma(\cdot, \cdot) \end{aligned}$$

□

Lemma 4.3. *Consider two (j, J) -holomorphic immersions $f_1: \Sigma \rightarrow M$ and $f_2: \Sigma \rightarrow M$. If ϕ is a diffeomorphism $\Sigma \rightarrow \Sigma$ such that $f_1 \circ \phi = f_2$, then $\phi \in \text{Aut}(\Sigma, j)$.*

Proof.

$$df_1 \circ d\phi \circ j = df_2 \circ j = J \circ df_2 = J \circ df_1 \circ d\phi = df_1 \circ j \circ d\phi.$$

Since f_1 is an immersion, we conclude $d\phi \circ j = j \circ d\phi$.

□

Lemma 4.4. *Let J be an almost complex structure on M , and $f: \Sigma \rightarrow M$ an immersed (j, J) -holomorphic curve. Assume that J is ω -compatible. If v is not everywhere tangent to $f(\Sigma)$, then $\omega_{\text{red}}^D([v], [\tilde{J}v]) \neq 0$.*

Proof. Decompose $v = \xi_v + \tau_v$, with ξ_v everywhere ω -orthogonal to Σ and τ_v everywhere tangent to Σ . By assumption $\xi_v \neq 0$. Then $\tilde{J}v = \tilde{J}\xi_v + \tilde{J}\tau_v$ with $\tilde{J}\xi_v \neq 0$. By Lemma A.7, $[\tilde{J}v] = [\tilde{J}\xi_v] \neq 0$. Thus, since J is ω -tamed and σ is an area form, $\omega_{\text{red}}^D([v], [\tilde{J}v]) = \omega_{\text{red}}^D([\xi_v], [\tilde{J}\xi_v]) = \int_{\Sigma} \omega(\xi_v, J(\xi_v)) \sigma \neq 0$.

□

Proof of Corollary 1.4. $\text{Ham}^J(M, \omega)$ is a closed subgroup of $\text{Ham}(M, \omega)$. (Notice that a diffeomorphism $\phi: M \rightarrow M$ is J -holomorphic iff ϕ^{-1} is.) Since J is regular for the projection map $p_A: \mathcal{M}_i(A, \Sigma, \mathcal{J}) \rightarrow \mathcal{J}$, the space $\mathcal{M}_e(A, \Sigma, J)$ of embedded (j, J) -holomorphic Σ -curves in a homology class $A \in H_2(M, \mathbb{Z})$ is a finite-dimensional manifold [9, Thm 3.1.5]. By Lemma 4.1, the group $\text{Ham}^J(M, \omega)$ acts on $\mathcal{M}_e(A, \Sigma, J)$ by composition on the left. Since the action of $\text{Ham}^J(M, \omega)$ on $C^\infty(\Sigma, M)$ preserves both $\omega_{(\Sigma, \sigma)}^D$ and the almost complex structure $\tilde{J}: \text{TC}^\infty(\Sigma, M) \rightarrow \text{TC}^\infty(\Sigma, M)$ induced from $J: \text{T}M \rightarrow \text{T}M$, it also preserves the map defined by

$$\tilde{g}(\tau_1, \tau_2) = \omega_{(\Sigma, \sigma)}^D(\tau_1, \tilde{J}\tau_2).$$

By Lemma A.5, the manifold $\mathcal{M}_e(A, \Sigma, J)$ is closed under \tilde{J} , hence by Lemma A.4, the restriction of $\omega_{(\Sigma, \sigma)}^D$ to $\mathcal{M}_e(A, \Sigma, J)$ is non-degenerate, so the map \tilde{g} restricts to a metric on $\mathcal{M}_e(A, \Sigma, J)$, and $\text{Ham}^J(M, \omega)$ acts on the moduli space as a subgroup of the isometry group. Since the action of the isometry group on a finite-dimensional manifold is proper and $\text{Ham}^J(M, \omega)$ is a closed subgroup, every orbit of $\text{Ham}^J(M, \omega)$ is an embedded submanifold of $\mathcal{M}_e(A, \Sigma, J)$.

Let \mathcal{N} be an orbit of $\text{Ham}^J(M, \omega)$ through an embedded (j, J) -holomorphic curve $f: \Sigma \rightarrow M$ for which $f^*\omega = \sigma$. By Lemma 4.1, the orbit $\mathcal{N} \subset \mathcal{M}_e(A, \Sigma, J) \cap \mathcal{S}_e(\Sigma, \sigma)$. Thus, by Lemma 2.5 and Lemma 3.3, the reductions ω_{red}^D and $\omega_{\mathcal{S}_i(\Sigma)}^{\text{red}}$ are well defined on the quotient \mathcal{N} modulo $\text{Symp}(\Sigma, \sigma)$. By Lemmata 4.2 and 4.3, \mathcal{N} modulo $\text{Aut}(\Sigma, j)$ is the same as \mathcal{N} modulo $\text{Symp}(\Sigma, \sigma)$. By Lemma 4.4, the form ω_{red}^D restricted to \mathcal{N} modulo $\text{Aut}(\Sigma, j)$ is non-degenerate. By Lemma 3.2, the form $\omega_{\mathcal{S}_i(\Sigma)}^{\text{red}}$ coincides with the form $2\omega_{\text{red}}^D$ on the quotient. \square

Remark 4.5 For examples of regular integrable compatible almost complex structures, we look at Kähler manifolds whose automorphism groups act transitively. By [9, Proposition 7.4.3], if (M, ω_0, J_0) is a compact Kähler manifold and G is a Lie group that acts transitively on M by holomorphic diffeomorphisms, then J_0 is regular for every $A \in H_2(M, \mathbb{Z})$. This applies, e.g., when $M = \mathbb{C}P^n$, ω_0 the Fubini-Study form, J_0 the standard complex structure on $\mathbb{C}P^n$, and G is the automorphism group $\text{PSL}(n+1)$, \odot

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A Appendix : a symplectic form and compatible almost complex structures on the space of immersed surfaces

To prove the closedness part of Proposition 2.1, we first show the following claim.

Claim A.1. *For $\tau_1, \tau_2: \Sigma \rightarrow f^*(TM)$, we have*

$$\omega(\tau_1, \tau_2) \sigma = \frac{1}{2} \iota_{(\tau_1, 0), (\tau_2, 0)} (\text{ev}^*(\omega)) \wedge (\pi_\Sigma^*(\sigma)),$$

where $\pi_\Sigma: C^\infty(\Sigma, M) \times \Sigma \rightarrow \Sigma$ is the projection onto the second coordinate.

Proof. Indeed,

$$\iota_{(\tau_1, 0), (\tau_2, 0)} (\text{ev}^*(\omega)) \wedge (\pi_\Sigma^*(\sigma))(\cdot, \cdot) = 2\omega(\tau_1, \tau_2)\sigma(\cdot, \cdot) + 2\omega(\tau_1, df(\cdot)) \wedge \sigma(d\pi_\Sigma(\tau_2, 0), \cdot).$$

Notice that the second summand in the right-hand term always vanishes. \square

Proof of the closedness part of Proposition 2.1. We will show that for any two surfaces R_1 and R_2 in $\mathcal{S}_i(\Sigma)$, that are homologous relative to a common boundary ∂R ,

$$\int_{R_1} \omega_{(\Sigma, \sigma)}^D = \int_{R_2} \omega_{(\Sigma, \sigma)}^D. \quad (\text{A.14})$$

See also [6, Prop. 12] for a proof that $d\omega_{(\Sigma, \sigma)}^D = 0$ in the Convenient Setup.

Let R_1 and R_2 be two surfaces that are homologous relative to a common boundary ∂R . By the above claim,

$$\int_{R_i} \omega^D_{(\Sigma, \sigma)} = \frac{1}{2} \int_{R_i \times \{x\}} \left(\int_{\{f\} \times \Sigma} (\text{ev}^*(\omega)) \wedge (\pi_\Sigma^*(\sigma)) \right) = \frac{1}{2} \int_{R_i \times \Sigma} (\text{ev}^*(\omega)) \wedge (\pi_\Sigma^*(\sigma)).$$

Since $R_1 \times \Sigma$ is homologous to $R_2 \times \Sigma$ relative to the boundary $\partial R \times \Sigma$, and $(\text{ev}^*(\omega)) \wedge (\pi_\Sigma^*(\sigma))$ is closed, we have that:

$$\int_{R_1 \times \Sigma} (\text{ev}^*(\omega)) \wedge (\pi_\Sigma^*(\sigma)) = \int_{R_2 \times \Sigma} (\text{ev}^*(\omega)) \wedge (\pi_\Sigma^*(\sigma)).$$

Therefore, we get (A.14). □

Compatible almost complex structures

An *almost complex structure* on a manifold M is an automorphism of the tangent bundle,

$$J: TM \rightarrow TM,$$

such that $J^2 = -\text{Id}$. The pair (M, J) is called an *almost complex manifold*.

An almost complex structure is *integrable* if it is induced from a complex manifold structure. In dimension two any almost complex manifold is integrable (see, e.g., [8, Th. 4.16]). In higher dimensions this is not true [1].

Definition A.2 Let J be an almost complex structure on M . We define a map

$$\tilde{J}: \text{TC}^\infty(\Sigma, M) \rightarrow \text{TC}^\infty(\Sigma, M)$$

as follows: for $\tau: \Sigma \rightarrow f^*(TM)$, the vector $\tilde{J}(\tau)$ is the section $\hat{J} \circ \tau$, where \hat{J} is the map defined by the commutative diagram

$$\begin{array}{ccc} f^*(TM) & \xrightarrow{\hat{J}} & f^*(TM) \\ \downarrow & \circlearrowleft & \downarrow \\ TM & \xrightarrow{J} & TM \end{array}$$

⊗

Due to the properties of the almost complex structure J , the map \tilde{J} is an automorphism and $\tilde{J}^2 = -\text{Id}$.

Claim A.3. *Let J be an almost complex structure on M . Then \tilde{J} is an almost complex structure on $\text{C}^\infty(\Sigma, M)$.*

An almost complex structure J on M is *tamed* by a symplectic form ω if $\omega_x(v, Jv) > 0$ for all non-zero $v \in T_x M$. If, in addition, $\omega_x(Jv, Jw) = \omega_x(v, w)$ for all $v, w \in T_x M$, we say that J is ω -compatible. The space $\mathcal{J} = \mathcal{J}(M, \omega)$ of ω -compatible almost complex structures is non-empty and contractible, in particular path-connected [8, Prop. 4.1].

Lemma A.4. *Let J be an almost complex structure on M .*

1. If J is ω -tamed, then the induced almost complex structure \tilde{J} on the space of immersions $\Sigma \rightarrow M$ is $\omega^D_{(\Sigma, \sigma)}$ -tamed.
2. If J is ω -compatible, then \tilde{J} is $\omega^D_{(\Sigma, \sigma)}$ -compatible.

Proof. 1. For a non-zero vector field $\tau: \Sigma \rightarrow f^*TM$,

$$\omega^D_{(\Sigma, \sigma)}_f(\tau, \tilde{J}(\tau)) := \int_{\Sigma} \omega(\tau, J(\tau)) \sigma > 0,$$

the last inequality is since J is ω -tamed, f is an immersion, and σ is an area-form.

2. For $\tau_1, \tau_2: \Sigma \rightarrow f^*TM$,

$$\omega^D_{(\Sigma, \sigma)}_f(\tilde{J}(\tau_1), \tilde{J}(\tau_2)) := \int_{\Sigma} \omega(J(\tau_1), J(\tau_2)) \sigma = \int_{\Sigma} \omega(\tau_1, \tau_2) \sigma = \omega^D_{(\Sigma, \sigma)}_f(\tau_1, \tau_2),$$

since J is ω -compatible. □

Proof of the non-degeneracy part of Proposition 2.1. It follows from part (1) of Lemma A.4, and the fact that the space of ω -tamed structures is not empty. □

Fix $\Sigma = (\Sigma, j)$. A smooth (C^∞) curve $f: \Sigma \rightarrow M$ is called *J-holomorphic* if the differential df is a complex linear map between the fibers $T_p(\Sigma) \rightarrow T_{f(p)}(M)$ for all $p \in \Sigma$, i.e.

$$df_p \circ j_p = J_{f(p)} \circ df_p.$$

A *J-holomorphic* curve is *simple* if it cannot be factored through a branched covering of Σ . The moduli space $\mathcal{M}_i(A, \Sigma, J)$ of simple immersed *J-holomorphic* Σ -curves in a homology class $A \in H_2(M, \mathbb{Z})$. We look at almost complex structures that are regular for the projection map

$$p_A: \mathcal{M}_i(A, \Sigma, \mathcal{J}) \rightarrow \mathcal{J};$$

for such a J , the space $\mathcal{M}_i(A, \Sigma, J)$ is a finite-dimensional manifold [9, Thm. 3.1.5].

Lemma A.5. *If J is an integrable almost complex structure on M that is regular for A , and $f: \Sigma \rightarrow M$ is a J -holomorphic map in A , then for $\tau \in T_f \mathcal{M}(A, \Sigma, J)$, the vector $J \circ \tau$ is also in $T_f \mathcal{M}(A, \Sigma, J)$.*

For proof, see, e.g., [2, Lem. 3.3].

As a result of Lemma A.5 and Lemma A.4, we get the following proposition.

Proposition A.6. *If J is an integrable almost complex structure on M that is compatible with ω and regular for A , then $\omega^D_{(\Sigma, \sigma)}$ restricted to $\mathcal{M}_i(A, \Sigma, J)$ is symplectic.*

Lemma A.7. *Let J be an almost complex structure on M . Assume that $f: \Sigma \rightarrow M$ is J -holomorphic. Then, at $x \in \Sigma$,*

1. if $\tau_x \in df_x(T_x \Sigma)$ then $J_{f(x)}(\tau_x) \in df_x(T_x \Sigma)$;
2. if J is ω -compatible and ϕ_x is ω -orthogonal to $df_x(T_x \Sigma)$, then so is $J_{f(x)}(\phi_x)$.

Proof. 1. By assumption $\tau_x = df_x(\alpha)$ for $\alpha \in T_x \Sigma$. Hence, since f is J -holomorphic,

$$J_{f(x)}(\tau_x) = J_{f(x)}(df_x(\alpha)) = df_x(j_x \alpha).$$

2. By the previous item, $J_{f(x)}(df_x(T_x \Sigma)) \subseteq df_x(T_x \Sigma)$, hence, since $J^2 = -\text{Id}$,

$$J_{f(x)}(df_x(T_x \Sigma)) = df_x(T_x \Sigma).$$

Let $\tau_x \in df_x(T_x \Sigma)$, then there exists $\tau'_x \in df_x(T_x \Sigma)$ such that $\tau_x = J_{f(x)}(\tau'_x)$. By assumption, $\omega(\phi_x, \tau'_x) = 0$. Thus

$$\omega(J_{f(x)}(\phi_x), \tau_x) = \omega(J_{f(x)}(\phi_x), J_{f(x)}(\tau'_x)) = \omega(\phi_x, \tau'_x) = 0.$$

□

Corollary A.8. *Let J be an ω -tamed almost complex structure on M . Assume that $f: \Sigma \rightarrow M$ is J -holomorphic. Then every $\mu \in T_f(C^\infty(\Sigma, M))$ can be uniquely decomposed as*

$$\mu = \mu' + \mu'',$$

where $\mu'(x) \in df_x(T_x \Sigma)$ at every $x \in \Sigma$, and $\mu''(x)$ is ω -orthogonal to $df_x(T_x \Sigma)$ at every $x \in \Sigma$.

Proof. At $x \in \Sigma$, if $v \in W_x = df_x(T_x \Sigma)$, then $J(v) \in W_x$ (by part (1) of Lemma A.7). Since J is ω -tamed, if $v \neq 0$, $\omega(v, J(v)) > 0$, hence $0 \neq v \in W_x$ is not in W_x^ω . Thus $W_x \cap W_x^\omega = \{0\}$. Since $\dim W_x^\omega = \dim M - \dim W_x$, we deduce that $T_{f(x)} M = W_x \oplus W_x^\omega$.

To conclude the corollary, recall that if a bundle $E \rightarrow B$ equals the direct sum of sub-bundles $E_1 \rightarrow B$ and $E_2 \rightarrow B$, then the space of sections of E equals the direct sum of the space of sections of E_1 and the space of sections of E_2 .

□

Remark A.9 Fix a symplectic form σ on and a complex structure j that is σ -compatible on Σ . Given a symplectic embedding $f: (\sigma, \Sigma) \rightarrow (M, \omega)$ there is a $J \in \mathcal{J}(M, \omega)$ such that f is (j, J) -holomorphic. Define $J|_{T(f(\Sigma))}$ such that f is holomorphic. Extend it to a compatible fiberwise complex structure on the symplectic vector bundle $TM|_{f(\Sigma)}$. Then extend it to a compatible almost complex structure on M . See [8, Section 2.6]. ⊗

References

- [1] E. Calabi: Constructions and properties of some 6-dimensional almost complex manifolds, *Trans. Am. Math. Soc.* **87**(1958), 407–438.
- [2] J. Coffey, L. Kessler, A. Pelayo: Symplectic geometry on moduli spaces of J -holomorphic curves, *Ann. Glob. Anal. Geom.*(2011). doi: 10.1007/s10455-011-9281-1
- [3] S. K. Donaldson: Moment maps and diffeomorphisms, *Asian J. Math.***3** (1999), no. 1, 1–16.
- [4] M. Gromov: Pseudo holomorphic curves in symplectic manifolds, *Invent. Math.* **82** (1985), 307–347.

- [5] A. Kriegl and P. W. Michor: *The convenient setting of global analysis*, Mathematical Surveys and Monographs, Vol. **53**, American Mathematical Society, Providence, RI (1997).
- [6] B. Lee: Geometric Structures on Spaces of Weighted Submanifolds, *SIGMA* **5** (2009), 099, (46 pages).
- [7] J-H. Lee: Symplectic geometry on symplectic knot spaces. *N. Y. J. Math.* **13** (2007), 17–31.
- [8] D. McDuff and D. Salamon: *Introduction to Symplectic Topology*, 2nd Ed., Oxford University Press, Oxford (1998).
- [9] D. McDuff and D. Salamon: *J-Holomorphic Curves and Symplectic Topology*, American Mathematical Society, Providence, RI (2004).
- [10] J. Moser: On the volume elements on a manifold, *Trans. Amer. Math. Soc.* **120** (1965) 286–294.
- [11] J. P. Ortega and T. S. Ratiu: *The optimal momentum map*. In: Geometry, Mechanics, and Dynamics, pp 329–362. Springer, New York (2002).
- [12] S. Sternberg: On minimal coupling, and the symplectic mechanics of a classical particle in the presence of a Yang-Mills field, *Proc. Nat. Acad. Sci. U.S.A.* **74** (1977), 5253–4.
- [13] A. Weinstein: Symplectic manifolds and their Lagrangian submanifolds, *Advances in Math.* **6** (1971), 329–346.

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